MATH 465 Fall 2024 Self Assessment

Calculus

Concepts: continuous, differentiable, gradient, chain rule, Taylor series.

1. Consider $f : \mathbb{R}^d \to \mathbb{R}, x \mapsto \frac{1}{2}x^\top Ax + b^\top x + c$, where $A \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d, c \in \mathbb{R}$. Is f continuous, differentiable? If f is differentiable, what is the gradient of f?

2. Consider $f : \mathbb{R}^d \to \mathbb{R}, x \mapsto \frac{1}{2}\|Ax + b\|_2^2$, where $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$. Is f continuous, differentiable? If f is differentiable, what is the gradient of f?

3. Consider differentiable $f : \mathbb{R}^m \to \mathbb{R}$ and define $g : \mathbb{R}^d \to \mathbb{R}, x \mapsto f(Ax+b)$, where $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$, compute ∇g.

4. If $f : \mathbb{R} \to \mathbb{R}$ is smooth, what is the Taylor series of f at a?

5. If $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable, what is the necessary condition for x^* to be the minimizer/maximizer of f?

Linear Algebra

Concepts: rank, trace, orthonomality, positivity, vector norm, Frobenius norm, spectral norm, eigenvalue decomposition, singular value decomposition (optional).

1. What is the tightest upper bound on $|x^\top y|$ in terms of Euclidean norms of x, y?

2. Let matrices A, B have the same dimensions. Show that the trace of the matrix $AB^\top$ is the same as $\sum_{i,j} A_{i,j}B_{i,j}$.

3. Why is it that $A^\top A + I$ is invertible for any matrix A? (I is the identity matrix.)

4. Does any matrix have an eigenvalue decomposition? Does any symmetric matrix have an eigenvalue decomposition? Is eigenvalue decomposition unique?

5. If $x^\top A^\top Ax = 0$, what can we say about x?

6. If U is an orthogonal matrix, argue that $\|Ux\|_2^2 = \|x\|_2^2$.
7. Argue that $\|A\|_F = \text{tr}(A^\top A)$ and $\|A\|_F \leq \|A\|_2$.
8. If A, B are matrices whose columns are respectively $\{a_i\}_i, \{b_i\}_i$ show that $AB^\top = \sum_i a_i b_i^\top$.
9. Express $\|aa^\top\|_F^2$ in terms of the Euclidean length of a.

   (below are optional.)
10. Express $\|A^{-1}\|_2$ in terms of the singular values of A.
11. If $A = USV^\top$ is the full SVD of A, what is the full SVDs of $A^\top A, AA^\top$? What is the eigenvalue decomposition of $A^\top A, AA^\top$.
12. If $A = USV^\top$ is the full SVD of A, how can you read off the rank and nullity of A from just S?
13. If A is symmetric, how are its eigenvalue decomposition and singular value decompostion related?

Probability

Concepts: independence, variance, expectation.

1. Let $p(x, y)$ be the joint pdf of random variables X, Y, give the expression of $p_1(x), p_2(y)$ marginal pdf of X, Y.
2. If X, Y are independent, how is p related to p_1, p_2?
3. If X, Y are independent, does $\mathbb{E}[f(x)g(y)] = \mathbb{E}[f(x)]\mathbb{E}[g(y)]$ holds for arbitrary functions?
4. What is $\mathbb{P}\{\mathcal{E}_1 \cup \mathcal{E}_2\}$ if the events $\mathcal{E}_1, \mathcal{E}_2$ are independent?
5. If X, Y, Z are independent, what can we say about $\mathbb{E}[X + Y + Z]$? What is they are not independent?
6. Give a sufficient condition for when the variance of $X + Y$ equals the sum of the variances of X and Y. Provide a proof of your claim.
7. Let X follows a standard Gaussian distribution in $\mathbb{R}^d$, U be an orthogonal matrix, show that UX follows the same distribution as X.
8. If $X_1, \cdots, X_n$ are independent identically distributed samples drawn from a distribution with finite variance, what can we say about the distribution of their average $\frac{1}{n}\sum_{i=1}^n X_i$?