MATH 465/466 Self Assessment

If you are unfamiliar with most of the concepts listed below or unsure about the answers to the majority of the questions, it is recommended that you complete the prerequisite courses before enrolling in Math 465/466.

Calculus

Concepts: continuous, differentiable, gradient, chain rule, Taylor series.

1. Consider $f : \mathbb{R}^d \to \mathbb{R}, x \mapsto \frac{1}{2}x^\top Ax + b^\top x + c$, where $A \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d, c \in \mathbb{R}$. Is f continuous, differentiable? If f is twice-differentiable, what is the gradient and Hessian of f?

2. Consider $f : \mathbb{R}^d \to \mathbb{R}, x \mapsto \frac{1}{2}\|Ax + b\|_2^2$, where $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$. Is f continuous, differentiable? If f is twice-differentiable, what is the gradient and Hessian of f?

3. Consider differentiable $f : \mathbb{R}^m \to \mathbb{R}$ and define $g : \mathbb{R}^d \to \mathbb{R}, x \mapsto f(Ax+b)$, where $A \in \mathbb{R}^{m\times d}, b \in \mathbb{R}^m$, compute ∇g.

4. Taylor's Theorem, Peano form of the remainder, Lagrange form of the remainder and integral form of the remainder.

Linear Algebra

Concepts: rank, trace, vector norm; linear dependence/indepedence, basis, dimension; orthogonal vector, orthogonal bases, orthogonal subspaces; eigenvalue, eigenvectors, eigen decomposition, positive (semi-)definite matrix; (Optional: Frobenius norm, singular value decomposition, condition number, spectral norm.)

1. If A, B are matrices whose columns are respectively $\{a_i\}_i, \{b_i\}_i$ show that $AB^\top = \sum_i a_i b_i^\top$.
2. Let matrices A, B have the same dimensions. Show that the trace of the matrix $\mathrm{tr}(AB^\top) = \sum_{i,j} A_{i,j} B_{i,j}$.
3. What is the tightest upper bound on $|x^\top y|$ in terms of Euclidean norms of x, y?
4. If U is an orthogonal matrix, argue that $\|Ux\|_2^2 = \|x\|_2^2$.
5. Does any matrix have an eigenvalue decomposition? Does any symmetric matrix have an eigenvalue decomposition? Is eigenvalue decomposition unique?
6. Why is it that $A^\top A + I$ is invertible for any matrix A? (I is the identity matrix.)
7. If $x^\top A^\top A x = 0$, what can we say about x?

   (below are optional.)
8. Argue that $\|A\|_F = \mathrm{tr}(A^\top A)$ and $\|A\|_F \leq \|A\|_2$.
9. Express $\|aa^\top\|_F^2$ in terms of the Euclidean length of a.
10. Express $\|A^{-1}\|_2$ in terms of the singular values of A.
11. If $A = USV^\top$ is the full SVD of A, what is the full SVDs of $A^\top A, AA^\top$? What is the eigenvalue decomposition of $A^\top A, AA^\top$.
12. If $A = USV^\top$ is the full SVD of A, how can you read off the rank and nullity of A from just S?

Probability

Concepts: independence, expectation, variance, central limit theorem; Gaussian distribution, binomial distribution.

1. Let $p(x, y)$ be the joint pdf of random variables X, Y, give the expression of $p_1(x), p_2(y)$ marginal pdf of X, Y. If X, Y are independent, how is p related to p_1, p_2?

2. When does $\mathbb{E}[f(x)g(y)] = \mathbb{E}[f(x)]\mathbb{E}[g(y)]$ holds for arbitrary functions? When does $\mathbb{E}[f(x) + g(y)] = \mathbb{E}[f(x)] + \mathbb{E}[g(y)]$?

3. Let X be a random vector in $\mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ be a deterministic matrix and $b \in \mathbb{R}^m$ be a deterministic vector. If the mean and covariance matrix of X is μ and Σ, what is the mean and covariance matrix of $AX + b$? If X follows Gaussian distribution, what distribution does $AX + b$ follows?

4. If $X_1, \cdots, X_n$ are independent identically distributed samples drawn from a distribution with finite variance, what can we say about the distribution of their average $\frac{1}{n}\sum_{i=1}^{n} X_i$?